QUANTUM INVARIANT MEASURES

NICOLAI RESHETIKHIN AND MILEN YAKIMOV

Abstract. We derive an explicit expression for the Haar integral on the quantized algebra of regular functions $\mathbb{C}_q[K]$ on the compact real form K of an arbitrary simply connected complex simple algebraic group G. This is done in terms of the irreducible $*$-representations of the Hopf $*$-algebra $\mathbb{C}_q[K]$. Quantum analogs of the measures on the symplectic leaves of the standard Poisson structure on K which are (almost) invariant under the dressing action of the dual Poisson algebraic group K^* are also obtained. They are related to the notion of quantum traces for representations of Hopf algebras. As an application we define and compute explicitly quantum analogs of Harish-Chandra c-functions associated to the elements of the Weyl group of G.

1. Introduction

Let G be a simply connected complex simple algebraic group. The cocommutative Hopf algebra $\mathbb{C}[G]$ of regular functions on G has a standard quantization, denoted by $\mathbb{C}_q[G]$ and called quantized algebra of regular functions on G. It is a Hopf subalgebra of the dual Hopf algebra of the standard quantized universal enveloping algebra $U_q\mathfrak{g}$. Let K denote a compact real form of G. The complex conjugation in the algebra $\mathbb{C}[K](=\mathbb{C}[G])$ can be deformed to a conjugate linear antiisomorphism $*$ of $\mathbb{C}_q[G]$. This gives rise to a Hopf $*$-algebra $(\mathbb{C}_q[G], *)$ called quantized algebra of regular functions on K which will be denoted by $\mathbb{C}_q[K]$.

The Hopf algebra $\mathbb{C}_q[K]$ is known [1] to have a unique Haar functional $H\colon \mathbb{C}_q[K] \to \mathbb{C}$ normalized by $H(1) = 1$. It is known by a quantum analog of the Schur orthogonality relations. At the same time an analog of the classical expression for the bi-invariant functional on $\mathbb{C}[K]$ as an integral over K with respect to the Haar measure was found only in the case of SU_2, [16]. The first result which we obtain in this paper is a representation for the Haar integral on $\mathbb{C}_q[K]$ of this type in the general case.

Let us first note that the quantum analog of the set of points on K is the set of irreducible $*$-representations of the Hopf $*$-algebra $\mathbb{C}_q[K]$. Its representations were classified by Soibelman [14] and can be nicely described by a version of Kirillov–Kostant orbit method. Fix a maximal torus T of K. Let $G = KAN$ be the related Iwasawa decomposition of G. The group K has a standard Poisson structure making it a real Poisson algebraic group which is the semiclassical structure of the deformation of $\mathbb{C}[K]$ to $\mathbb{C}_q[K]$. The double and dual Poisson algebraic groups of K are isomorphic to G and AN as real algebraic groups, respectively. The dressing action of AN on K is global and is explicitly given by the rule [9, 14]

$$\delta_{an}(k) \quad \text{for} \quad a \in A, n \in N, k \in K \quad \text{is such that} \quad ank = (\delta_{an}(k))\, a_1 n_1 \tag{1.1}$$

The first author was partially supported by NSF grant DMS96-03239.
The second author was supported by NSF grants DMS94-00097 and DMS96-03239.

for some $a_1 \in A, n_1 \in N$ (see [13, 9] for general facts about the dressing action). Let us choose for each element w of the Weyl group W of G a representative $\dot{w}$ in the normalizer of A in K. The orbits of the dressing action of AN on K (symplectic leaves of K) are $\mathcal{S}_w.t$ where $w \in W$, $t \in T$ and $\mathcal{S}_w$ denotes the orbit of $\dot{w}$. The disjoint union $\sqcup_{t \in T} \mathcal{S}_w.t$ is the Bruhat cell $K \cap B\dot{w}B$ where B is the Borel subgroup $B = TAN$ of G. Soibelman proved that the leaves $\mathcal{S}_w.t$ are deformed to a set $\pi_{w,t}$ of (unequivalent) irreducible $*$-representations of the Hopf $*$-algebra $\mathbb{C}_q[K]$. Up to an equivalence they exhaust all such representations of $\mathbb{C}_q[K]$.

Our result on the Haar integral on $\mathbb{C}_q[K]$ expresses it as an integral over the maximal torus T of K of the traces of the representations $\pi_{w_\circ,t}$ for the maximal length element $w_\circ$ of W. In other words these are the irreducible $*$-representations of $\mathbb{C}_q[K]$ corresponding to the symplectic leaves in the maximal Bruhat cell of K. This result is derived in Section 5. It is particularly suited for obtaining integral expressions for quantum spherical functions. This will be discussed in a future publication.

For each $w \in W$ denote $N_w = N \cap wN_- w^{-1}$ and $N_w^+ = N \cap wNw^{-1}$ where N_- is the opposite to N unipotent subgroup of G. Our next result is a quantum analog of the Haar measures on the unipotent groups N_w. The symplectic leaf $\mathcal{S}_w.t$, considered as an AN-homogeneous space via the dressing action, is isomorphic to

$$S_w.t = AN/AN_w^+. \tag{1.2}$$

The quotient AN/AN_w^+ does not have a left invariant measure because the ratio of the corresponding modular functions is not equal to 1, see [3]. Using the factorization $AN = N_w AN_w^+$ we can identify $AN/AN_w^+ \cong N_w$ which induces a measure on the symplectic leaf (1.2) from the Haar measure on N_w. The resulting measure transforms under the action of AN by the following multiplicative character of AN

$$\chi(an) = a^{2(\rho - w\rho)}, \quad a \in A, n \in N. \tag{1.3}$$

The dressing action of $AN = K^*$ on the symplectic leaf $\mathcal{S}_w.t$ of K induces an action of K^* on the space of functions on $\mathcal{S}_w.t$. The latter transforms in the quantum situation to an action of $\mathbb{C}_q[G]$ on the space of linear operators in the Hilbert space completion $\overline{V}_{w,t}$ of the representation space of $\pi_{w,t}$. It coincides with the standard adjoint action

$$c.L = \sum \pi_{w,t}(c_{(1)}) L \pi_{w,t}(S(c_{(2)})). \tag{1.4}$$

(Here and later we use the standard notation for the comultiplication in a Hopf algebra $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$.) Let us also note that $\mathbb{C}_q[K]$ acts by bounded operators in all of its $*$-representations and thus in particular in $\overline{V}_{w,t}$.

The standard trace in $\overline{V}_{w,t}$ is not a homomorphism from the space of trace class operators in $\overline{V}_{w,t}$ with the adjoint $\mathbb{C}_q[G]$-action (1.4) to the 1-dimensional representation of $\mathbb{C}_q[G]$ determined by its counit. After Reshetikhin and Turaev such a homomorphism, from possibly a "deformation" of the space of trace class operators, is called a quantum trace for the Hopf algebra module under consideration. We define a space $\mathcal{B}_1^q(\overline{V}_{w,t})$ of "quantum" trace class operators in $\overline{V}_{w,t}$, stable under the adjoint $\mathbb{C}_q[G]$-action (1.4), and construct a homomorphism from it to the 1-dimensional representation of $\mathbb{C}_q[G]$ determined by a multiplicative character of it which is a deformation of the character (1.3). Such homomorphisms, to be called quantum quasi-traces, are treated in Section 6 where we also study some of their

properties. They are quantum analogs of the invariant measures on the unipotent groups N_w and the almost AN-invariant measures on the symplectic leaves $\mathcal{S}_w.t$.

Section 7 contains an application to quantum analogs of Harish-Chandra c-functions related to the elements of the Weyl group of G. They are constructed by the help of the quantum quasi-traces from Section 6 and are explicitly computed by a q-analog of the original Harish-Chandra formula. In the quantum situation the role of the factorization formulas for the groups N_w as products of 1-dimensional unipotent subgroups is played by tensor product formulas for the representations $\pi_{w,t}$ [14, 7]. In a forthcoming publication we will discuss the relation between the quantum c-functions and the asymptotics of quantum spherical functions at infinity which is similar to the one in the classical case.

Sections 2 and 3 review some standard facts about quantized universal enveloping algebras, quantized function algebras, and their representations. Section 4 deals with a family of elements of $\mathbb{C}_q[K]$ which enter in all formulas for quantum invariant functionals derived in this paper.

2. Preliminaries on quantized enveloping algebras

2.1. **Root data.** Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l with Cartan matrix (a_{ij}). Denote by $(.,.)$ the invariant inner product on $\mathfrak{g}$ for which the square length of a minimal root equals 2 in the resulting identification $\mathfrak{h}^* \cong \mathfrak{h}$ for a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The sets of simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$ will be denoted by $\{\alpha_i\}_{i=1}^l$, $\{\alpha_i^\vee\}_{i=1}^l$, and $\{\omega_i\}_{i=1}^l$, respectively. Let P, Q, and $Q^\vee$, denote the weight, root, and coroot lattices of $\mathfrak{g}$. Denote by Δ, Δ_+, Δ_-, and P_+ the sets of roots, positive/negative roots, and dominant weights of $\mathfrak{g}$. Set $Q_+ = \{\sum m_i \alpha_i\}$ and $Q_+^\vee = \{\sum m_i \alpha_i^\vee\}$, $m_i \in \mathbb{N}$.

Recall that there exists a unique set of relatively prime positive integers $\{d_i\}_{i=1}^l$ for which the matrix $(d_i a_{ij})$ is symmetric and for it

$$(\alpha_i, \alpha_j) = d_i a_{ij}.$$

The Weyl group of $\mathfrak{g}$ will be denoted by W. The simple reflections in W will be denoted by s_i and the maximal length element in W by $w_\circ$.

2.2. **Definition of $U_q\mathfrak{g}$.** Throughout this paper we will assume that q is a real number different from ± 1 and 0. The adjoint rational form of the quantized universal enveloping algebra $U_q\mathfrak{g}$ of $\mathfrak{g}$ is generated by $K_i^{\pm 1}$, and $X_i^\pm$, $i = 1, \dots, l$, subject to the relations

$$K_i^{-1} K_i = K_i K_i^{-1} = 1, \; K_i K_j = K_j K_i,$$
$$K_i X_j^\pm K_i^{-1} = q_i^{a_{ij}} X_j^\pm,$$
$$X_i^+ X_j^- - X_j^- X_i^+ = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{r=0}^{1-a_{ij}} \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} (X_i^\pm)^r X_j^\pm (X_i^\pm)^{1-a_{ij}-r} = 0, \; i \neq j.$$

It is a Hopf algebra with comultiplication given by

$$\Delta(K_i) = K_i \otimes K_i,$$
$$\Delta(X_i^+) = X_i^+ \otimes K_i + 1 \otimes X_i^+,$$
$$\Delta(X_i^-) = X_i^- \otimes 1 + K_i^{-1} \otimes X_i^-,$$

antipode and counit given by

$$S(K_i) = K_i^{-1},\ S(X_i^+) = -X_i^+ K_i^{-1},\ S(X_i^-) = -K_i X_i^-,$$
$$\epsilon(K_i),\ \epsilon(X_i^\pm) = 0$$

where $q_i = q^{d_i}$. As usual q-integers, q-factorials, and q-binomial coefficients are denoted by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},\ [n]_q! = [1]_q \dots [n]_q,\ \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q}{[m]_q [n-m]_q}$$

for $n, m \in \mathbb{N}$ and $m \leq n$.

The conjugate linear antiisomorphism $*$ of $U_q\mathfrak{g}$ defined on its generators by

$$K_i^* = K_i,\ (X_i^+)^* = X_i^- K_i,\ (X_i^-)^* = K_i^{-1} X_i^- \tag{2.5}$$

equips $U_q\mathfrak{g}$ with a structure of a Hopf $*$-algebra. In the limit $q \to 1$ the involution $*$ recovers the Cartan (anti)involution (conjugate linear antiisomorphism of order 2) of $\mathfrak{g}$ associated to its compact real form $\mathfrak{k}$. For the definition and properties of Hopf $*$-algebras we refer to [7, pp. 95–97] and [1, pp. 117–118].

For $i = 1, \dots, l$ the Hopf subalgebra of $U_q\mathfrak{g}$ generated by K_i and $X_i^\pm$ will be denoted by $U_{q_i}\mathfrak{g}_i$. It is naturally isomorphic to $U_{q_i}\mathfrak{sl}_2$. The canonical embedding $U_q\mathfrak{sl}_2 \cong U_{q_i}\mathfrak{g}_i \hookrightarrow U_q\mathfrak{g}$ will be denoted by φ_i. Recall that a $U_q\mathfrak{g}$-module is called *integrable* if the subalgebras $U_{q_i}\mathfrak{g}_i$ act locally finitely.

The subalgebras of $U_q\mathfrak{g}$ generated by $\{K_i\}_{i=1}^l$, $\{X_i^+\}_{i=1}^l$, and $\{X_i^-\}_{i=1}^l$ will be denoted by U_0, U^+, and U^-, respectively. Clearly U_0 is a commutative Hopf subalgebra of $U_q\mathfrak{g}$ isomorphic to the group algebra of the lattice Q equipped with the standard structure of a cocommutative Hopf algebra.

2.3. **Quantum Weyl group.** Let $\mathcal{B}_\mathfrak{g}$ denote the (generalized) braid group associated to the Coxeter group W with generators T_i corresponding to the simple reflections $s_i \in W$. For any integrable $U_q\mathfrak{g}$-module V one can define an action of $\mathcal{B}_\mathfrak{g}$ on V. It is given by [10]

$$T_i = \sum_{a,b,c \in \mathbb{N}} (-1)^b q_i^{ac-b} (X_i^+)^{(a)} (X_i^-)^{(b)} (X_i^+)^{(c)}$$

where

$$(X_i^\pm)^{(n)} = \frac{X_i^\pm}{[n]_{q_i}}.$$

In the case of the adjoint representation of $U_q\mathfrak{g}$ this gives an action of the braid group $\mathcal{B}_\mathfrak{g}$ on $U_q\mathfrak{g}$. The explicit action of T_i on the generators K_j, $X_j^{\pm}$ of $U_q\mathfrak{g}$ is

$$T_i(X_i^+) = -X_i^- K_i,\; T_i(X_i^-) = -K_i^{-1}X_i^+,\; T_i(K_j) = K_j K_i^{-a_{ij}},$$
$$T_i(X_j^+) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{-r} (X_i^+)^{(-a_{ij}-r)} X_j^+ (X_i^+)^{(r)} \quad \text{if} \quad i \neq j,$$
$$T_i(X_j^-) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{r} (X_i^-)^{(r)} X_j^- (X_i^-)^{(-a_{ij}-r)} \quad \text{if} \quad i \neq j.$$

The defined actions of $\mathcal{B}_\mathfrak{g}$ are compatible in the sense that for any integrable $U_q\mathfrak{g}$-module V

$$T_i.xv = (T_i x).T_i v,\ \forall x \in U_q\mathfrak{g}, \quad v \in V.$$

Recall that there exists a canonical section $T\colon W \to \mathcal{B}_\mathfrak{g}$ of the natural projection $\mathcal{B}_\mathfrak{g} \to W$ (where $T_i \mapsto s_i$). If

$$w = s_{i_1} \dots s_{i_n}$$

is a reduced decomposition of $w \in W$ then the image T_w of w in $\mathcal{B}_\mathfrak{g}$ is defined by

$$T_w = T_{i_1} \dots T_{i_n}.$$

It does not depend on the choice of a reduced decomposition.

The weight subspaces of a U_0-module (in particular of a $U_q\mathfrak{g}$-module) V are defined by

$$V_\lambda = \{v \in V \mid K_i.v = q^{(\lambda,\alpha_i)} v\}, \quad \lambda \in P.$$

The elements of $\mathcal{B}_\mathfrak{g}$ preserve the weight space decomposition of an integrable $U_q\mathfrak{g}$-module, in particular

$$T_w V_\lambda = V_{w\lambda}.$$

2.4. ***R*-matrix.** Put

$$U_k^{\pm} = \bigoplus_{\substack{\lambda \in \pm Q_+, \\ |(\lambda,\rho^\vee)| \geq k}} U_\lambda^{\pm}, \quad k \in \mathbb{N} \tag{2.6}$$

where $\rho^\vee$ is the half-sum of positive coroots of $\mathfrak{g}$. Denote by $U_+ \widehat{\otimes} U_-$ the completion of the vector space $U_+ \widehat{\otimes} U_-$ according to the descending sequence of vector spaces

$$\left(U_k^+ \otimes U^-\right) \oplus \left(U^+ \otimes U_k^-\right).$$

Any element of the completion $U_+ \widehat{\otimes} U_-$ acts in the tensor product of two finite dimensional $U_q\mathfrak{g}$-modules.

Recall that a representation V of $U_q\mathfrak{g}$ is called a type 1 representation if it is a direct sum of its weight subspaces. For a pair (V_1, V_2) of type 1 $U_q\mathfrak{g}$-modules define the linear operator $\Psi_{V_1,V_2}\colon V_1 \otimes V_2 \to V_1 \otimes V_2$ by

$$\Psi_{V_1,V_2}(v_1 \otimes v_2) = q^{(\lambda,\mu)} v_1 \otimes v_2 \quad \text{if} \quad v_1 \in (V_1)_\lambda,\ v_2 \in (V_2)_\mu.$$

Denote also by $\sigma\colon V_1 \otimes V_2 \to V_2 \otimes V_1$ the flip operator

$$\sigma(v_1 \otimes v_2) = v_2 \otimes v_1.$$

There exists [10, 7] a unique element $R \in U^+ \widehat{\otimes} U^-$, called *a quasi R-matrix* for $U_q\mathfrak{g}$, normalized by

$$R - 1 \in U_1^+ \widehat{\otimes} U_1^-$$

such that for any pair (V_1, V_2) of finite dimensional $U_q\mathfrak{g}$-modules of type 1 the composition

$$\sigma \circ \Psi_{V_1,V_2} \circ R : V_1 \otimes V_2 \to V_2 \otimes V_1 \tag{2.7}$$

defines an isomorphism of $U_q\mathfrak{g}$-modules.

For any pair (V_1, V_2) of finite dimensional $U_q\mathfrak{g}$-modules and an element $w \in W$ the actions of $T_w \in \mathcal{B}_\mathfrak{g}$ on V_1, V_2, and $V_1 \otimes V_2$, to be denoted by T_{w,V_1}, T_{w,V_2}, and $T_{w,V_1 \otimes V_2}$, are related as follows. There exists a unique element $R^w \in U^+ \widehat{\otimes} U^-$ which does not depend on V_1 and V_2 such that

$$T_{w,V_1 \otimes V_2} = R^w \left(T_{w,V_1} \otimes T_{w,V_2} \right). \tag{2.8}$$

As the quasi R-matrix R, R^w satisfies

$$R^w - 1 \in U_1^+ \widehat{\otimes} U_1^-. \tag{2.9}$$

The element $R^{w_\circ}$ associated to the maximal element $w_\circ$ of W is equal to the quasi R-matrix R.

3. Quantized algebras of functions

3.1. **Quantized algebras of regular functions.** Let G be a connected, simply connected, complex simple algebraic group and $\mathfrak{g} = \mathrm{Lie} G$. The finite dimensional, $U_q\mathfrak{g}$-modules of type 1 form a quasitensor category. Hence their matrix coefficients form a Hopf subalgebra of the Hopf dual $(U_q\mathfrak{g})^*$ of $U_q\mathfrak{g}$. It is called the quantized algebra of regular functions on G and is denoted by $\mathbb{C}_q[G]$.

Every finite dimensional type 1 $U_q\mathfrak{g}$-module is a direct sum of irreducible type 1 $U_q\mathfrak{g}$-modules. The latter are highest weight modules with highest weights $\Lambda \in P_+$ (the corresponding module will be denoted by $L(\Lambda)$). The matrix coefficient of $L(\Lambda)$ associated to $v \in L(\Lambda)$ and $l \in L(\Lambda)^*$ will be denoted by $c^\Lambda_{l,v}$:

$$c^\Lambda_{l,v} \in \mathbb{C}_q[G],\ c^\Lambda_{l,v}(x) = \langle l, x.v \rangle.$$

The above implies

$$\mathbb{C}_q[G] = \mathrm{span}\{ c^\Lambda_{l,v} \mid \Lambda \in P_+,\ v \in L(\Lambda),\ l \in L(\Lambda)^* \}.$$

The $*$-involution in $U_q\mathfrak{g}$ induces a structure of Hopf $*$-algebra on $\mathbb{C}_q[G]$ by

$$\langle \xi^*, x \rangle = \overline{\langle \xi, S(x)^* \rangle}, \quad \xi \in \mathbb{C}_q[G],\ x \in U_q\mathfrak{g}. \tag{3.10}$$

The resulting Hopf $*$-algebra $(\mathbb{C}_q[G], *)$ is called quantized algebra of regular functions on the compact real form K of G and is denoted by $\mathbb{C}_q[K]$.

The inclusions $\varphi_i \colon U_{q_i}\mathfrak{g}_i \hookrightarrow U_q\mathfrak{g}$ induce surjective homomorphisms $\varphi_i^* \colon (\mathbb{C}_q[G], *) \to (\mathbb{C}_{q_i}[G_i], *)$ where G_i is the subgroup of G isomorphic to SL_2 with tangent Lie algebra $\mathfrak{g}_i$ generated by the root vectors of $\pm\alpha_i$.

We finish this subsection with a simple fact on the explicit structure of the Hopf $*$-algebra $\mathbb{C}_q[K]$ (see, for instance, [1, Proposition 13.1.3]).

Recall that $L(\Lambda)^* \cong L(-w_\circ \Lambda)$ and if we fix these isomorphisms, we can consider any $v \in L(\Lambda)$, $l \in L(\Lambda)^*$ as elements of $L(-w_\circ\Lambda)^*$, $L(-w_\circ\Lambda)$, respectively. Recall that any module $L(\Lambda)$ can be equipped with a unique (up to a constant) inner product which turns it into a $(U_q\mathfrak{g}, *)$ $*$-representation.

Lemma 3.1. *(i) The comultiplication, the counit, and the antipode of* $\mathbb{C}_q[G]$ *are given by*

$$\Delta(c^{\Lambda}_{l,v}) = \sum_j c^{\Lambda}_{l,v_j} \otimes c^{\Lambda}_{l_j,v}, \tag{3.11}$$

$$\epsilon(c^{\Lambda}_{l,v}) = \langle l, v\rangle,\ S(c^{\Lambda}_{l,v}) = c^{-w_\circ\Lambda}_{v,l} \tag{3.12}$$

where in (3.11) $(\{v_j\}, \{l_j\})$ *is an arbitrary pair of dual bases of* $L(\Lambda)$ *and* $L(\Lambda)^*$.

(ii) Fix an orthonormal basis $\{v_i\}$ *of* $L(\Lambda)$ *equipped with an invariant inner product as above and a dual basis* $\{l_j\}$ *of* $L(\Lambda)^*$. *The action of the* $*$*-involution* (3.10) *on the corresponding elements of* $\mathbb{C}_q[G]$ *is given by*

$$(c^{\Lambda}_{l_i,v_j})^* = (c^{-w_\circ\Lambda}_{v_i,l_j}). \tag{3.13}$$

3.2. **Quantized algebra of continuous functions of** K**.** Let G be a complex simple algebraic group as in the previous subsection and K be its compact real form. The quantized algebra of continuous functions $\mathcal{C}_q(K)$ on K is by definition the C^*-completion of the $*$-algebra $\mathbb{C}_q[K]$ with respect to the norm

$$\|f\| = \sup_\eta \|\eta(f)\|, \quad f \in \mathbb{C}_q[K] \tag{3.14}$$

where η runs through all $*$-representations of $\mathbb{C}_q[K]$.

The fact that for any $*$-representation η of $\mathbb{C}_q[K]$ $\eta(f)$ is a bounded operator and that the supremum in (3.14) is finite for all $f \in \mathbb{C}_q[G]$ follows from the following identity in $\mathbb{C}_q[K]$

$$\sum_j c^{\Lambda}_{l_j,v_i}(c^{\Lambda}_{l_j,v_i})^* = 1$$

where $\{v_i\}$ and $\{l_j\}$ are dual bases of $L(\Lambda)$ and $L(\Lambda)^*$ as in part (ii) of Lemma 3.1, see [1, eq. (13) p. 452].

The C^*-algebras $\mathcal{C}_q(K)$ posses natural structures of compact matrix quantum groups in the sense of Woronowicz [18], see [1, Section 13.3].

3.3. $\mathbb{C}_q[SU_2]$**.** The $U_q\mathfrak{sl}_2$-module $L(\omega_1)$ has a basis in which the operators K_1, $X_1^{\pm}$ act by

$$K_1 \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1}\end{pmatrix}, \quad X_1^+ \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \quad X_1^- \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}.$$

The corresponding matrix coefficients $c_{ij} \in \mathbb{C}_q[SL_2]$ $i,j = 1,2$ generate $\mathbb{C}_q[SL_2]$. More precisely:

Lemma 3.2. *The Hopf algebra* $\mathbb{C}_q[SL_2]$ *is isomorphic to the algebra generated by* c_{ij}, $i,j = 1,2$, *subject to the relations*

$$\begin{aligned} c_{11}c_{12} &= q^{-1}c_{12}c_{11}, & c_{11}c_{21} &= q^{-1}c_{21}c_{11}, \\ c_{12}c_{22} &= q^{-1}c_{22}c_{12}, & c_{21}c_{22} &= q^{-1}c_{22}c_{21}, \\ c_{12}c_{21} &= c_{21}c_{12}, & c_{11}c_{22} - c_{22}c_{11} &= (q^{-1}-q)c_{12}c_{21} \\ & & c_{11}c_{22} - q^{-1}c_{12}c_{21} &= 1. \end{aligned}$$

In these generators the comultiplication, the counit, the antipode, and the $*$-involution of $\mathbb{C}_q[SU_2]$ are given by

$$\Delta(c_{ij}) = \sum_{k=1,2} c_{ik} \otimes c_{kj},\ \epsilon(c_{ij}) = \delta_{ij},$$
$$S(c_{11}) = c_{22},\ S(c_{22}) = c_{11},\ S(c_{12}) = -qc_{12},\ S(c_{21}) = -q^{-1}c_{21},$$
$$c_{11}^* = c_{22},\ c_{21}^* = -qc_{12}.$$

A proof of Lemma 3.2 can be found, for instance, in [7, Example 2.3.3 and Theorem 3.0.1].

Let $q \in \mathbb{R}$, $q > 1$. The Hopf $*$-algebra $\mathbb{C}_q[SU_2]$ has an infinite dimensional $*$-representation π on $l^2(\mathbb{N})$ given by the following action of its generators c_{ij}, $i, j = 1, 2$ (see [14, 7])

$$\pi(c_{12})e_k = q^{-k-1}e_k, \quad \pi(c_{11})e_k = \sqrt{1-q^{-2k}}e_{k-1}, \tag{3.15}$$

$$\pi(c_{21})e_k = -q^{-k}e_k, \quad \pi(c_{22})e_k = \sqrt{1-q^{-2k-2}}e_{k+1} \tag{3.16}$$

where $e_{-1} := 0$.

3.4. **Irreducible star representations of $\mathbb{C}_q[K]$.** The group of multiplicative characters of the Hopf algebra $C_q[G]$ is isomorphic to the complex torus $(\mathbb{C}^\times)^l$, see [4, Theorem 3.3] and [6, Section 10.3.8] in the case when q is an indeterminate. The character corresponding to the l-tuple $t = (t_1, \dots, t_l) \in (\mathbb{C}^\times)^l$ is given by

$$\chi_t(c^\Lambda_{l,v}) = \prod_{i=1}^{l} t_i^{(\lambda, \alpha_i^\vee)} \langle l, v\rangle = \prod_{i=1}^{l} t_i^{(\lambda, \alpha_i^\vee)} \epsilon(c^\Lambda_{l,v}),\ v \in L(\Lambda)_\lambda. \tag{3.17}$$

The unitary ones among these are the ones corresponding to the real torus $(S^1)^l = \{(t_1, \dots, t_l) \in (\mathbb{C}^\times)^l \mid |t_i| = 1\}$.

From now on we will assume that $q \in \mathbb{R}$, $q > 1$. Denote by π_i the $*$-representation of $(\mathbb{C}_{q_i}[G_i], *) \cong \mathbb{C}_{q_i}[SU_2]$ given by (3.15)–(3.16). The $*$-representation of $\mathbb{C}_q[K] \cong (\mathbb{C}_q[G], *)$ induced from it by the homomorphism $\varphi_i^* : (\mathbb{C}_q[G], *) \to (\mathbb{C}_{q_i}[G_i], *)$ will be denoted by π_{s_i}. (Recall that s_i denotes the simple reflection in the Weyl group W of $\mathfrak{g}$ corresponding to the root α_i.)

The irreducible $*$-representations of the Hopf $*$-algebra $\mathbb{C}_q[K]$ were classified by Soibelman [14], see also the book [7] for an exposition.

Theorem 3.3. *(i) For any reduced decomposition $w = s_{i_1} \dots s_{i_n}$ of an element w of W and any $t \in (S^1)^l$ the tensor product*

$$\pi_{w,t} = \pi_{s_{i_1}} \otimes \dots \otimes \pi_{s_{i_n}} \otimes \chi_t \tag{3.18}$$

is an irreducible $*$-representation of $\mathbb{C}_q[K]$.

(ii) Up to an equivalence the representation $\pi_{w,t}$ does not depend on the choice of reduced decomposition of w.

(iii) Every irreducible $*$-representation of $\mathbb{C}_q[G]$ is isomorphic to some $\pi_{w,t}$.

Denote by $V_{w,t}$ the representation space of $\pi_{w,t}$ equipped with the Hermitian inner product from Theorem 3.3. The Hilbert space completion of $V_{w,t}$ with respect to it will be denoted by $\overline{V}_{w,t}$. Then:

The representations $\pi_{w,t}$ naturally induce irreducible representations of the C^*-algebra $\mathcal{C}_q(K)$, $\pi_{w,t} \colon \mathcal{C}_q(K) \to \mathcal{B}(\overline{V}_{w,t})$. The latter exhaust all irreducible representations of $\mathcal{C}_q(K)$ up to a unitary equivalence.

Each module $V_{w,t}$ has a natural orthonormal basis

$$e_{k_1,\dots,k_n} = e_{k_1} \otimes \dots \otimes e_{k_n} \otimes 1, \quad n = l(w),\ k_1, \dots, k_n \in \mathbb{N} \tag{3.19}$$

induced from the orthonormal basis $\{e_k\}$ of the $\mathbb{C}_q[SU_2]$-module V defined by (3.15)–(3.16). Here 1 denotes a (fixed) vector of the 1-dimensional representation of $\mathbb{C}_q[G]$ corresponding to χ_t.

For an element w of the Weyl group W denote by I_w the $*$-ideal of $\mathbb{C}_q[K]$ generated by

$$c^{\Lambda}_{l,v_\Lambda} \quad \text{such that} \quad \Lambda \in P_+,\ \langle l, U^+ T_w.v_\Lambda\rangle = 0 \tag{3.20}$$

where v_Λ denotes a highest weight vector of $L(\Lambda)$.

The annihilation ideals of the representations $\pi_{w,t}$ are contained in I_w [14, 7]:

$$\ker \pi_{w,t} \subset I_w. \tag{3.21}$$

4. A family of elements $a_{\Lambda,w} \in \mathbb{C}_q[K]$

4.1. **Definitions.** For a dominant integral weight $\Lambda \in P_+$ and a highest weight vector v_Λ of $L(\Lambda)$ denote by $l_{\Lambda,w}$ the unique element of $L(\Lambda)^*_{-w\Lambda}$ such that

$$\langle l_{\Lambda,w}, T_w v_\Lambda\rangle = 1.$$

(The uniqueness follows from the fact that $\dim L(\Lambda)_{w\Lambda} = 1$.)

Define

$$a_{\Lambda,w} = c^{\Lambda}_{l_{\Lambda,w},v_\Lambda}. \tag{4.22}$$

Note that $a_{\Lambda,w}$ does not depend on the choice of highest weight vector v_Λ of $L(\Lambda)$.

The $*$-subalgebras of $\mathbb{C}_q[K]$ generated by $a_{\Lambda,w}$ played an important role in Soibelman's classification of the irreducible $*$-representations of $\mathbb{C}_q[K]$, see Theorem 3.3. Most of the results in this subsection are due to Soibelman [14]. We include their proofs since [14] does not assume the normalization made in the definition of $a_{\Lambda,w}$.

Properties (2.8) and (2.9) of the elements $R^w \in U^+ \widehat{\otimes} U^-$ allow to write $l_{\Lambda,w}$ and thus $a_{\Lambda,w}$ slightly more explicitly. Let $l_\Lambda = l_{\Lambda,1}$, i.e. let $l_\Lambda \in L(\Lambda)^*_{-\Lambda}$ be the unique element such that

$$\langle l_\Lambda, v_\Lambda\rangle = 1.$$

Then (2.8), (2.9) imply

$$l_{\Lambda,w} = T_w l_\Lambda$$

and thus

$$a_{\Lambda,w} = c^{\Lambda}_{T_w l_\Lambda, v_\Lambda}. \tag{4.23}$$

Proposition 4.1. *(i) The elements $a_{\Lambda,w}$, $a^*_{\Lambda,w} \in \mathbb{C}_q[K]$, $\Lambda \in P_+$ are normal modulo I_w :*

$$a_{\Lambda,w} c^{\Lambda'}_{l,v} - q^{(\Lambda,\lambda')-(w\Lambda,\mu')} c^{\Lambda'}_{l,v} a_{\Lambda,w} \in I_w, \tag{4.24}$$

$$a^*_{\Lambda,w} c^{\Lambda'}_{l,v} - q^{(\Lambda,\lambda')-(w\Lambda,\mu')} c^{\Lambda'}_{l,v} a^*_{\Lambda,w} \in I_w, \tag{4.25}$$

for $v \in L(\Lambda')_{\lambda'}$, $l \in L(\Lambda')^*_{-\mu'}$.

(ii) The images of $\{a_{\Lambda,w}, a^*_{\Lambda,w}\}_{\Lambda \in P_+}$ in $\mathbb{C}_q[K]/I_w$ generate a commutative subalgebra. More precisely the following identity holds in $\mathbb{C}_q[K]$

$$a_{\Lambda_1,w} a_{\Lambda_2,w} = a_{\Lambda_1+\Lambda_2,w}, \quad \forall \Lambda_1, \Lambda_2 \in P_+. \tag{4.26}$$

Proofs of Proposition 4.1 can be found in [14, 7]. The property (4.24) follows from the existence of a quasi R-matrix for $U_q\mathfrak{g}$, see (2.7). Eq. (4.25) follows from (4.24), Lemma 3.1, and the fact that the ideals I_w are stable under the $*$-involution. The first statement in part (ii) is a direct consequence of part (i). The second statement in (ii) follows from the existence of the element $R^w \in U_1^+ \widehat{\otimes} U_1^-$ with the properties (2.8), (2.9) and the fact that $v_{\Lambda_1} \otimes v_{\Lambda_2} \in L(\Lambda_1) \otimes L(\Lambda_2)$ generates a submodule isomorphic to $L(\Lambda_1 + \Lambda_2)$.

4.2. **The action of $a_{\Lambda,w}$ in $V_{w,t}$.**

Lemma 4.2. *Let $w, w' \in W$ be such that $w = s_i w'$ and $l(w) = l(w') + 1$ for some simple reflection $s_i \in W$. Then*

$$\Delta(a_{\Lambda,w}) - c^{\Lambda}_{l_{\Lambda,w}, T_{w'} v_\Lambda} \otimes a_{\Lambda,w'}.$$

Proof. According to (3.11) $\Delta(a_{\Lambda,w})$ is given by

$$\Delta(a_{\Lambda,w}) = \sum_j c^{\Lambda}_{l_{\Lambda,w}, v_j} \otimes c^{\Lambda}_{l_j, v_\Lambda}$$

where $(\{v_j\}, \{l_j\})$ is a pair of dual bases of $L(\Lambda)$ and $L(\Lambda)^*$ consisting of weight vectors ($v_j \in L(\Lambda)_{\lambda_j}$, $l_j \in L(\Lambda)_{-\lambda_j}$, $\lambda_j \in P$). The definition (3.20) of $I_{w'}$ implies

$$c^{\Lambda}_{l_j, v_\Lambda} \in I_{w'} \quad \text{if} \quad \lambda_j \notin w\Lambda + Q_+. \tag{4.27}$$

The map $\varphi_i^* \colon \mathbb{C}_q[G] \to \mathbb{C}_{q_i}[G_i]$ acts on the matrix coefficients of a $U_q\mathfrak{g}$-module by restricting the module to $U_{q_i}\mathfrak{g}_i$. Since $w = s_i w'$ and $l(w) = l(w') + 1$

$$w^{-1}\alpha_i^\vee \in -Q_+^\vee.$$

If Λ is a dominant weight, then

$$\langle \Lambda, w^{-1}\alpha_i^\vee \rangle \leq 0 \quad \text{and thus} \quad \langle w\Lambda, \alpha_i^\vee \rangle \leq 0.$$

Hence $T_w v_\Lambda$ is a lowest weight vector for the $U_{q_i}\mathfrak{g}_i$-submodule of $L(\Lambda)$ generated by $T_w v_\Lambda$. The corresponding $U_{q_i}\mathfrak{g}_i$-highest weight vector is $T_{w'} v_\Lambda$ and

$$c^{\Lambda}_{l_j, v_\Lambda} \in (\varphi_i^*)^{-1}(T_{s_i}) \quad \text{if} \quad \lambda_j \notin \{w\Lambda, w\Lambda + \alpha_i, \ldots, w'\Lambda\}. \tag{4.28}$$

The lemma now follows from (4.27) and (4.28). $\square$

For an element $w \in W$ and a reduced decomposition $w = s_{i_1} \ldots s_{i_n}$ of it denote

$$w_j = s_{i_{j+1}} \ldots s_{i_n},\ j = 0, \ldots, n-1, \quad w_n = 1. \tag{4.29}$$

Proposition 4.3. *In the notation (4.29) the action of the elements $a_{\Lambda,w}$ in the module $V_{w,t}$ is given by*

$$\pi_{w,t}(a_{\Lambda,w}) = \bigotimes_{j=1}^{n} \pi_{s_{i_j}}(a_{(w_j\Lambda, \alpha_{i_j}^\vee)\omega_{i_j}, s_{i_j}}).\prod_{i=1}^{l} t_i^{(\Lambda,\alpha_i^\vee)}. \tag{4.30}$$

In the orthonormal basis $\{e_{k_1,\ldots,k_n}\}_{k_j=0}^{\infty}$, of $V_{w,t}$, see (3.19), the elements $a_{\Lambda,w}$ act diagonally by

$$\pi_{w,t}(a_{\Lambda,w}).e_{k_1,\ldots,k_n} = \prod_{j=1}^{n} q^{-(k_j+1)(w_j\Lambda, \alpha_{i_j})} \prod_{i=1}^{l} t_i^{(\Lambda,\alpha_i^\vee)} e_{k_1,\ldots,k_n}. \tag{4.31}$$

Formula (4.30) follows by induction from Lemma 4.2 and the definition (3.17) of the multiplicative characters χ_t of $\mathbb{C}_q[G]$. To prove (4.31) we first compute that in $\mathbb{C}_q[SL_2]$

$$a_{\omega_1,s_1} = -qc_{21} \tag{4.32}$$

(cf. Section 3.3) and then use (4.26) which implies $a_{m\omega_1,s_1} = (a_{\omega_1,s_1})^m$. We also use the identity $d_i\alpha_i^\vee = (\alpha_i,\alpha_i)\alpha_i^\vee/2 = \alpha_i$, see Section 2.1.

5. The Haar integral on $\mathcal{C}_q(K)$

5.1. **Definition and the Schur orthogonality relations.** Recall that a left invariant integral on a Hopf algebra A is a linear functional H on A satisfying

$$(\mathrm{id}\otimes H)\,(\Delta(a)) = H(a).1, \quad \forall a\in A. \tag{5.33}$$

Analogously is defined a right invariant integral. In the analytic setting a left Haar integral for a C^*-Hopf algebra A is a state H on A satisfying (5.33), see [18].

Proposition 5.1. *There exists a unique left invariant integral H on the Hopf algebra $\mathbb{C}_q[K]$ normalized by $H(1)=1$. It is also right invariant and can be uniquely extended to a bi-invariant Haar integral on $\mathcal{C}_q(K)$. It is given by a quantum version of the classical Schur orthogonality relations:*

$$H(c^{\Lambda}_{l,v}c^{\Lambda'}_{l',v'}) = \frac{\delta_{\Lambda,\Lambda'}\langle l,v'\rangle\langle l',v\rangle}{\sum_\lambda \dim L(\Lambda)_\lambda q^{2(\lambda,\rho)}}$$

or equivalently by

$$H(c^{\Lambda}_{l,v}) = \delta_{\Lambda,0}\langle l,v\rangle. \tag{5.34}$$

5.2. **Statement of the main result.**

Theorem 5.2. *The bi-invariant integral H on $\mathcal{C}_q(K)$ $(q\in\mathbb{R},\ q>1)$ is given in terms of the irreducible representations $\pi_{w,t}$ of $\mathcal{C}_q(K)$ by*

$$H(c) = \left(\prod_{\beta\in\Delta_+}(q^{(2\rho,\beta)}-1)\right)\int_{(S^1)^l}\mathrm{tr}_{\overline{V}_{w_\circ,t}}(\pi_{w_\circ,t}(a_{\rho,w_\circ}a^*_{\rho,w_\circ}c))dt \tag{5.35}$$

where $w_\circ$ is the maximal length element of the Weyl group W of $\mathfrak{g}$, ρ is the half sum of all positive roots of $\mathfrak{g}$, and dt is the invariant measure on the torus $(S^1)^l$ normalized by $\int_{(S^1)^l}dt = 1$.

In the special case of $K = SU_2$ Theorem 5.2 was established by Soibelman and Vaksman [16]. Similar formula is also known for quantum spheres [15, 17]. Theorem 5.2 answers Question 3 in [15].

Note that formula (4.31) implies that $\pi_{w,t}(a_{\rho,w_\circ}a^*_{\rho,w_\circ})$ is a trace class operator in $\overline{V}_{w,t}$. Since $\pi_{w,t}(c)$ is a bounded operator for $c\in\mathcal{C}_q(K)$, the product is also a trace class operator in $\overline{V}_{w,t}$ for all $c\in\mathcal{C}_q(K)$. From the definition (3.18) of $\pi_{w,t}$ it is also clear that

$$\mathrm{tr}_{\overline{V}_{w_\circ,t}}(\pi_{w_\circ,t}(a_{\rho,w_\circ}a^*_{\rho,w_\circ}c))$$

is a continuous function in $t\in(S^1)^l$ for a fixed $c\in\mathcal{C}_q(K)$ and that the r.h.s. of (5.35) defines a continuous linear functional on $\mathcal{C}_q(K)$.

By identifying $(S^1)^l \cong \{(t_1, \dots, t_l) \in \mathbb{C}^l \colon |t_i| = 1\}$, the normalized invariant measure on the torus $(S^1)^l$ is represented as

$$dt = \frac{1}{(2\pi i)^l} \frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_l}{t_l}.$$

In Sections 5.3 and 5.4 we show that the functional $\widetilde{H}$ on $\mathbb{C}_q[G]$ given by the right hand side of (5.35) satisfies

$$\widetilde{H}(c^{\Lambda}_{l,v}) = 0 \quad \text{if} \quad \Lambda \neq 0. \tag{5.36}$$

In Section 5.5 we check that it satisfies the normalization condition $\widetilde{H}(1) = 1$. Combined with (5.34) this proves Theorem 5.2.

5.3. **Proof of** (5.36)**: reduction to the rank 1 case.** Recall first the following simple characterization of $w_\circ \in W$.

Lemma 5.3. *The maximal length element $w_\circ \in W$ is the only element $w \in W$ that has a representation of the form $w = w' s_i$ with $l(w') = l(w) - 1$ for an arbitrary simple reflection s_i.*

Lemma 5.3 follows from the so called "deletion condition", see [5], and the property of $w_\circ$ that it is the only element $w \in W$ such that $w^{-1}(\alpha_i)$ is a negative root of $\mathfrak{g}$ for all simple roots α_i of $\mathfrak{g}$.

We show that (5.36) for $K = SU_2$ implies its validity in the general case. Let $\Lambda \in P_+$, $\Lambda \neq 0$. Equip $L(\Lambda)$ with a Hermitian inner product making it a $(U_q\mathfrak{g}, *)$ $*$-representation, recall (2.5). Denote

$$L_i = \{v \in L(\Lambda) \mid U_{q_i}\mathfrak{g}_i.v = 0\},\ i = 1, \dots, l.$$

Since $L(\Lambda)$ is an irreducible $U_q\mathfrak{g}$-module

$$\cap_{i=1}^l L_i = 0$$

and

$$L_1^\perp + \dots + L_2^\perp = (\cap_{i=1}^l L_i)^\perp = L(\Lambda).$$

Hence to show (5.36) it is sufficient to show that

$$\widetilde{H}(c^{\Lambda}_{l,v}) = 0 \quad \text{if} \quad v \in L_m^\perp \quad \text{for some} \quad m = 1, \dots, l. \tag{5.37}$$

Note that $L_m^\perp$ is the span of the nontrivial irreducible $U_{q_m}\mathfrak{g}_m$-submodules of $L(\Lambda)$.

Choose a reduced decomposition of $w_\circ$ of the form

$$w_\circ = s_{i_1} \dots s_{i_{n_\circ - 1}} s_m$$

and consider the corresponding model for the representation $\pi_{w_{n_\circ}, t}$

$$\pi_{w_{n_\circ}, t} \cong \pi_{s_{i_1}} \otimes \dots \otimes \pi_{i_{n_\circ - 1}} \otimes \pi_{s_m} \otimes \chi_t.$$

Taking trace over the component $\pi_{s_m} \otimes \chi_t$ of $\pi_{w_{n_\circ}, t}$ and using (3.11) and (4.30) we see that to prove (5.37) it is sufficient to prove that

$$\int_{(S^1)^l} \mathrm{tr}_{\overline{V}_{s_m}} (\pi_{s_m}(a_{\omega_m, s_m} a^*_{\omega_m, s_m} \varphi_m^*(c^{\Lambda}_{l', v}))) dt = 0 \quad \text{for all} \quad l' \in L(\Lambda)^*. \tag{5.38}$$

(Recall that by definition $(w_\circ)_{n_\circ} = 1$, see (4.29).) Since $v \in L_m^\perp$

$$\varphi_m^*(c^{\Lambda}_{l,v}) = \sum_p c^{p\omega_m}_{l_p, v_p}$$

with all $p > 0$. By appropriately breaking the integral (5.38) into a product of a 1-dimensional and an $(l-1)$-dimensional integrals one sees that (5.38) follows from (5.36) for $K = SU_2$.

5.4. **Proof of** (5.36)**: the case of** $\mathbb{C}_q[SU_2]$**.** Our proof in the rank 1 case is similar to the one from [16]. Lemma 3.2 implies that $\mathbb{C}_q[SU_2]$ is spanned by the elements

$$c_{11}^m c_{12}^p c_{21}^r \text{ and } c_{22}^m c_{12}^p c_{21}^r \quad \text{for} \quad m,p,r \in \mathbb{N}.$$

The Haar functional H acts on them by [1, Example 13.3.9]

$$H(c_{11}^m c_{12}^p c_{21}^r) = H(c_{22}^m c_{12}^p c_{21}^r) = \delta_{m,0}\delta_{p,r}\frac{(-q)^p(q^2-1)}{q^{2p+2}-1}\cdot$$

We check that the functional $\widetilde{H}$ has the same property. This implies (5.36) for $K = SU_2$.

Recall from (4.32) that $a_{\omega_1,s_1} = -q^{-1}c_{21}$ and thus $a^*_{\omega_1,s_1} = c_{12}$, see Lemma 3.2. Using (3.15)–(3.16) we compute

$$\begin{aligned}
\operatorname{tr}_{\overline{V}}(\pi(a_{\omega_1,s_1}a^*_{\omega_1,s_1}c_{ii}^m c_{12}^p c_{21}^r)) &= \delta_{m,0}\sum_{k=0}^{\infty} -q^{-1}.q^{-(k+1)(p+1)}.(-q^{-k})^{r+1} \\
&= \delta_{m,0}\frac{(-q)^r}{q^{p+r+2}-1}
\end{aligned}$$

for $i = 1, 2$. This gives

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{S^1}\operatorname{tr}_{\overline{V}_{s_1,t}}(\pi_{s_1,t}(a_{\omega_1,s_1}a^*_{\omega_1,s_1}c_{ii}^m c_{12}^p c_{21}^r))\frac{dt}{t} \\
&= \delta_{m,0}\frac{(-q)^r}{q^{p+r+2}-1}\int t^{r-p-1}dt = \delta_{m,0}\delta_{p,r}\frac{(-q)^p}{q^{2p+2}-1}
\end{aligned}$$

$(i = 1, 2)$ which shows that $\widetilde{H} = H$ in the case $K = SU_2$.

5.5. **Checking the normalization** $\widetilde{H}(1) = 1$**.** Let $w_\circ = s_{i_1}\dots s_{i_{n_\circ}}$ be a reduced decomposition of the maximal element of W. Using (4.31) and the notation (4.29) we compute

$$\begin{aligned}
\int_{(S^1)^l}\operatorname{tr}_{\overline{V}_{w_\circ,t}}(\pi_{w_\circ,t}(a_{\rho,w_\circ}a^*_{\rho,w_\circ}))dt &= \prod_{j=1}^{n_\circ}\left(\sum_{k_j=0}^{\infty} q_{i_j}^{-2(k_j+1)((w_\circ)_j\rho,\alpha_{i_j}^\vee)}\right) \\
&= \prod_{j=1}^{n_\circ}\frac{1}{1-q_{i_j}^{-(2\rho,(w_\circ)_j^{-1}\alpha_{i_j}^\vee)}}\cdot
\end{aligned}$$

Note that $q_i^{(\lambda,\alpha_i^\vee)} = q^{(\lambda,\alpha_i)}$ for all simple roots α_i of $\mathfrak{g}$. The set of elements $(w_\circ)_j^{-1}\alpha_{i_j} \in Q$, $j = 1,\dots,n_\circ$, coincides with the set of positive roots of $\mathfrak{g}$. This together with the definition of the functional $\widetilde{H}$ via the r.h.s. of (5.35) gives

$$\widetilde{H}(1) = 1.$$

5.6. **Semiclassical limit.** Here we explain the semiclassical analog of the integral formula from Theorem 5.2.

As earlier G denotes a complex simple algebraic group and K denotes a compact real form of G. For each element w of the Weyl group W of K choose a representative $\dot{w}$ of it in the normalizer of a fixed maximal torus T of K. Using the related Iwasawa decomposition of G, introduce the map

$$a_w \colon N \to A \quad \text{by} \quad \dot{w}^{-1} n \dot{w} = k_1 a_w(n) n_1, \quad k_1 \in K, n_1 \in N, \tag{5.39}$$

see for instance [8]. It can be pushed down to a well defined map from the symplectic leaf $\mathcal{S}_w$ to A

$$a_w(\delta_n \dot{w}) := a_w(n),\ n \in N.$$

We refer to the introduction for details on the dressing action of AN on K related to the standard Poisson structure on K.

The semiclassical analog of formula (5.35) is the following formula for the normalized Haar integral on K

$$H(f) = \left(\prod_{\beta \in \Delta_+} \frac{\pi}{(\rho, \beta)} \right) \int_{\mathcal{S}_{w_\circ} \times T} a_{w_\circ}(k)^{-2\rho} f(k.t) \mu_{w_\circ} dt, \quad f \in C(K). \tag{5.40}$$

Here $\mu_{w_\circ}$ denotes the Liouville volume form on the symplectic leaf $\mathcal{S}_{w_\circ}$ corresponding to the maximal element $w_\circ \in W$ and dt denotes the invariant measure on the torus T normalized by $\int_T dt = 1$. Recall that $\mathcal{S}_{w_\circ} \times T$ coincides with the maximal Bruhat cell of K.

Formula (5.40) can be easily proved following the idea of Sections 5.3–5.5 on the basis of the product formulas [14, 7] for the symplectic leaves $\mathcal{S}_w$ of K, $w \in W$

$$\mathcal{S}_w = \mathcal{S}_{s_{i_1}} \dots \mathcal{S}_{s_{i_n}} \tag{5.41}$$

where $s_{i_1} \dots s_{i_n}$ is a reduced decomposition of w.

The integral with respect to the symplectic measure on the leaf $\mathcal{S}_w.t$ is (up to a factor) a semiclassical limit of the trace in the module $\overline{V}_{w,t}$.

At the end we explain the connection between the functions $a_w^{-2\rho}$ on the leaves $\mathcal{S}_w$ and the operators $\pi_{w,t}(a_{\rho,w} a^*_{\rho,w})$ in $\overline{V}_{w,t}$. Let us consider the highest weight module $L(\Lambda)$ of $\mathfrak{g}$ with heighest weight Λ and the matrix coefficient

$$a_{\Lambda,w} \in \mathbb{C}[G], \quad a_{\Lambda,w}(g) = \langle l_{w,\Lambda}, g v_\Lambda \rangle,\ g \in G$$

where v_Λ is a highest weight vector of $L(\Lambda)$ and $l_{w,\Lambda} \in L(\Lambda)^*_{-w\Lambda}$ is normalized by $\langle l_{w,\Lambda}, \dot{w} v_\Lambda \rangle = 1$, cf. (4.22). It is easy to show that the restriction of $a_{\Lambda,w}$ to the symplectic leaf $\mathcal{S}_w$ coincides with $a_w^{-\Lambda}$

$$a_{\Lambda,w}|_{\mathcal{S}_w} = a_w^{-\Lambda}.$$

For $t \in T$ the functions

$$|a_{w,\rho}(k.t)|^2 = |a_{w,\rho}(k)|^2 = a_w(k)^{-2\rho},\ k \in \mathcal{S}_w$$

are semiclassical analogs of the linear operators $\pi_{w,t}(a_{w,\rho} a^*_{w,\rho})$ in $\overline{V}_{w,t}$.

6. Quantum quasi-traces of $V_{w,t}$

6.1. **Motivation.** Let A be a Hopf algebra and A^* be its dual Hopf algebra. Denote by A° the dual Hopf algebra of A equipped with the opposite comultiplication. Recall [2, 1] that the quantum double $\mathcal{D}(A)$ of A is isomorphic to $A \otimes A^\circ$ as a coalgebra and the following commutation relation holds in $\mathcal{D}(A)$

$$\xi a = \sum \langle \xi_{(1)}, a_{(3)} \rangle \xi_{(2)} a_{(2)} \langle S^{-1} \xi_{(3)}, a_{(1)} \rangle, \quad \xi \in A^*, a \in A. \tag{6.42}$$

Analogously to the classical situation one defines a quantum dressing action δ of A^* on A. Using the identification $\mathcal{D}(A) \cong A \otimes A^*$ as vector spaces, set

$$\delta_\xi a = (\mathrm{id} \otimes \epsilon)(\xi a).$$

In view of the commutation relation (6.42) it is explicitly given by

$$\delta_\xi a = \sum \langle \xi_{(1)}, a_{(3)} \rangle a_{(2)} \langle S^{-1} \xi_{(2)}, a_{(1)} \rangle.$$

It is dual to the standard adjoint action of A^* on itself

$$\mathrm{ad}_\xi \, \xi' = \sum \xi_{(1)} \xi' S(\xi_{(2)})$$

in the sence that

$$\langle \mathrm{ad}_\xi \, \xi', a \rangle = \langle \xi', \delta_{S(\xi)} a \rangle. \tag{6.43}$$

For any representation π of A^* in the vector space V the adjoint action of A^* on itself lifts to an action of A^* in the space of linear operators on V by

$$\mathrm{ad}_\xi \, L = \sum \pi(\xi_{(1)}) L \pi(S \xi_{(2)}). \tag{6.44}$$

Suppose that A^* is a deformation of the Poisson Hopf algebra $\mathbb{C}[F]$ of regular functions on a Poisson algebraic group F. According to Kirillov–Kostant orbit method philosophy an irreducible A^*-module V can be viewed as a quantization of a symplectic leaf $\mathcal{S}$ in F. The left action of A^* in the space of linear operators in V is a deformation of the Poisson $\mathbb{C}[F]$-module of functions on the leaf $\mathcal{S}$. At the same time the dual Poisson algebraic group F^* of F acts in the space of functions on $\mathcal{S}$ by the dressing action. The quantum analog of this action is the the adjoint action (6.44) of A^* in the space of linear operators in the A^*-module V. This leads to:

The quantum analog of a measure on the symplectic leaf $\mathcal{S}$ in the Poisson algebraic group F which is invariant up to a multiplicative character of F^* is a homomorphism from a subspace of linear operators in the A^*-module V, *equipped with the A^*-action* (6.44)*, to a 1-dimensional representation of A^*.*

In the next subsection we will develop this idea from a categorical point of view and relate it to the notion of quantum traces for A^*-modules. In analogy, the defined more general morphisms will be called quantum quasi-traces. Subsections 6.3 and 6.4 construct such morphisms for the irreducible $*$-representations of the quantized algebras of functions $(\mathbb{C}_q[G], *)$.

6.2. **Definitions.** Let $\mathcal{C}$ be a $\mathbb{C}$-linear, rigid, monoidal category with identity object $\mathbf{1}$. Recall that $\mathcal{C}$ is called balanced if for each object $V \in \mathrm{Ob}(\mathcal{C})$ there exists an isomorphism

$$b_V \colon V \xrightarrow{\cong} V^{**}$$

such that

$$b_{V_1} \otimes b_{V_2} = b_{V_1 \otimes V_2}, \tag{6.45}$$

$$b_{V^*} = (b_V^*)^{-1}, \tag{6.46}$$

$$b_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}. \tag{6.47}$$

Given a Hopf algebra C over the field $\mathbb{C}$ let rep_C denote the category of its finite dimensional modules equipped with the left dual object V^* of $V \in \mathrm{Ob}(\mathcal{C})$ defined by

$$\langle c.\xi, v\rangle = \langle \xi, S(a).v\rangle, \quad \xi \in V^*, v \in V. \tag{6.48}$$

The spaces $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, $V_1, V_2 \in \mathrm{Ob}(\mathcal{C})$ can be equipped with the canonical C-action

$$c.L = \sum \pi_{V_1}(c_{(1)}) L \pi_{V_2}(S(c_{(2)})), \quad L \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2). \tag{6.49}$$

Here the Hopf algebra C plays the role of the Hopf algebra A^* from the motivation in the previous subsection, cf. (6.44) and its derivation from the quantum dressing action.

Clearly

$$\mathrm{Hom}_{\mathbb{C}}(V_1, V_2) \cong V_2 \otimes V_1^*$$

as C-modules. In particular, for this action $\mathrm{Hom}_{\mathbb{C}}(V, \epsilon)$ is canonically isomorphic to V^* where, by abuse of notation, ϵ denotes the 1-dimensional representation of C defined by its counit.

Reshetikhin and Turaev [11] defined the following notion of quantum trace for a finite dimensional C-module V.

Definition 6.1. A *quantum trace* for a finite dimensional C-module V is a homomorphism

$$\mathrm{qtr}_V \colon \mathrm{End}_{\mathbb{C}}(V) \to \epsilon$$

of C-modules for the action of C on $\mathrm{End}_{\mathbb{C}}(V)$ defined in (6.49).

The pairing

$$\mathrm{End}_{\mathbb{C}}(V) \cong V \otimes V^* \to \mathbb{C}$$

is not a homomorphism of C-modules where $\mathbb{C}$ is given the structure of the C-module corresponding to the counit ϵ. At the same time the opposite pairing

$$V^* \otimes V \to \mathbb{C}$$

has this property. If rep_C is balanced each $V \in \mathrm{Ob}(\mathcal{C})$ has a quantum trace defined by the composition [11]

$$\mathrm{End}_{\mathbb{C}}(V) \cong V \otimes V^* \xrightarrow{b_V \otimes \mathrm{id}} V^{**} \otimes V^* \to \epsilon$$

or explicitly

$$\mathrm{qtr}_V(L) = \mathrm{tr}_V(b_V L), \; L \in \mathrm{End}(V).$$

Here b_V is considered as a linear endomorphism of V using the canonical identification of V and V^{**} as vector spaces.

The properties (6.45)–(6.46) of the balancing morphisms b_V imply the following properties of the quantum traces qtr_V

$$\mathrm{qtr}_{V_1\otimes V_2}(L_1\otimes L_2)=\mathrm{qtr}_{V_1}(L_1)\,\mathrm{qtr}_{V_2}(L_2), \tag{6.50}$$

$$\mathrm{qtr}_{V^*}(L^*)=\mathrm{qtr}_V(L) \tag{6.51}$$

for all $L_i\in\mathrm{End}_{\mathbb{C}}(V_i)$.

In [11, 12] it was proved that the category of finite dimensional type 1 $U_q\mathfrak{g}$-modules is balanced and this was used for constructing invariants of links and 3-dimensional manifolds.

We would like to incorporate in Definition 6.1 the possibility for an invariant up to a character "quantum measure", as explained in the previous section, and the general case of an infinite dimensional C-module V. We will restrict ourselves to representations of C $\pi\colon C\to\mathcal{B}(V)$ by bounded operators in a Hilbert space V and will call them *bounded* representations of C. The Hermitian inner product in V is not assumed to posses any invariance properties and the linear operators $\pi(c)$, $c\in C$, in V are not assumed to be uniformly bounded. The dual V^* of such a bounded representation $\pi\colon C\to\mathcal{B}(V)$ is defined in the Hilbert space V^* of bounded functionals on V by formula (6.48). Obviously it is again a bounded representation.

Definition 6.2. Two bounded representations of a Hopf algebra C $\pi_i\colon C\to\mathcal{B}(V_i)$ in the Hilbert spaces V_i, $i=1,2$, will be called weakly equivalent if V_i contain dense C-stable subspaces $W_i\subset V_i$ which are equivalent as C-modules.

The point here is that the equivalence can be given by an unbounded operator $b\colon W_1\xrightarrow{\cong}W_2$ which therefore does not extend to the full space V_1.

Definition 6.3. A bounded representation $\pi\colon C\to\mathcal{B}(V)$ of a Hopf algebra C in a Hilbert space V will be called quasi-balanced if there exists a multiplicative character χ of C for which V and $\chi\otimes V^{**}$ are weakly equivalent.

By abuse of notation we denote by χ the 1-dimensional C-module corresponding to the multiplicative character χ of C.

In other words the bounded C-module V is balanced if there exists an invertible linear operator b_V in V with dense domain and range such that $\mathrm{Dom}\,b_V$ is C-stable and

$$b_V\pi(c)=\sum\chi(c_{(1)})\pi(S^2(c_{(2)}))b_V,\quad \forall c\in C. \tag{6.52}$$

(Here we use the canonical identification of V^{**} and V as Hilbert spaces.)

Remark 6.4. Often V is the Hilbert space completion of a C-module W, equipped with a Hermitian inner product, which is a direct sum of mutually orthogonal finite dimensional submodules W_μ for a Hopf subalgebra B of C

$$W=\oplus_\mu W_\mu. \tag{6.53}$$

The restricted dual of such a module W with respect to the decomposition (6.53) as a direct sum of finite dimensional subspaces is naturally a C-module of the same type. The double restricted dual W^{**} of W is canonically isomorphic to W as a vector space.

If $W\cong\chi\otimes W^{**}$ as C-modules then the modules V and $\chi\otimes V^{**}$ are weakly equivalent and V is a quasi-balanced C-module.

Let $\pi\colon C \to \mathcal{B}(V)$ be a quasi-balanced representation as above. We call the subspace of the space of linear operators in V with dense domains

$$\mathcal{B}_1^q(V) := \mathcal{B}_1(V) b_V^{-1}$$

a space of *quantum trace class operators* in the C-module V. Here $\mathcal{B}_1(V)$ stands for the standard trace class in V. It is naturally a C-module by

$$c.L = \sum \pi(c_{(1)}) L \pi(S(a_{(c)}))$$

because C acts in V by bounded operators. The linear map $\operatorname{qtr}_V\colon \mathcal{B}_1^q(V) \to \mathbb{C}$ given by

$$\operatorname{qtr}_V(L) := \operatorname{tr}_V(L b_V)$$

is a well defined homomorphism of C-modules

$$\operatorname{qtr}_V\colon \mathcal{B}_1^q(V) \to \chi.$$

It will be called a quantum *quasi-trace* for the module V.

Remark 6.5. One can as well use the space

$$\widetilde{\mathcal{B}}_1^q(V) := b_V^{-1} \mathcal{B}_1(V)$$

instead of $\mathcal{B}_1^q(V)$. When b_V^{-1} is not defined on the full space V the composition $b_V^{-1} L_0$, $L_0 \in \mathcal{B}_1(V)$ need not have a dense domain in V. Because of this, it is convenient to use the space $\widetilde{\mathcal{B}}_1^q(V)$ only when b_V^{-1} has full domain. In that case the space $\widetilde{\mathcal{B}}_1^q(V)$ is also a C-module and the following map

$$\operatorname{qtr}_V\colon \widetilde{\mathcal{B}}_1^q(V) \to \chi, \; \operatorname{qtr}_V(L) := \operatorname{tr}(b_V L)$$

is a homomorphism of C-modules.

Remark 6.6. It is natural to look for a quasi-balancing map $b_V \in \operatorname{End}_{\mathbb{C}}(V)$ for a bounded representation $\pi\colon C \to \mathcal{B}(V)$ of the form

$$b_V = \pi(a_V)$$

for some $a_V \in C$. The definition (6.52) implies that such a map $\pi(a_V)$ provides a quasi-balancing endomorphism if $\pi(a_V)$ is an invertible linear operator in V with a dense range satisfying

$$a_V c - \sum \chi(c_{(1)}) S^2(c_{(2)}) a_V \in \operatorname{Ker}\pi, \quad \forall c \in C \tag{6.54}$$

for some multiplicative character χ of A.

Thus quasi-balancing of the modules of a Hopf algebra A is related to the properties of the the square of the antipode S of A. This is analogous to the usual case of balancing when $\chi = \epsilon$ and (6.54) reduces to

$$a_V c = S^2(c) a_V \in \operatorname{Ker}\pi, \quad \forall c \in C,$$

see [11].

Similarly

$$b_V = \pi_V(a_V)^{-1}$$

is a quasi-balancing map for the C-module V if $\pi_V(a_V)$ is an invertible operator in V with a dense range such that

$$c a_V - \sum \chi(c_{(1)}) a_V S^2(c_{(2)}) \in \operatorname{Ker}\pi, \quad \forall c \in C. \tag{6.55}$$

6.3. **Main construction.** In this subsection we construct quasi-balancing morphisms for the $\mathbb{C}_q[G]$-modules $V_{w,t}$. As was pointed out in Section 3.4 they are bounded $\mathbb{C}_q[G]$-modules in the terminology from the previous subsection.

Set

$$2\rho = \sum_{\alpha\in\Delta_+} \alpha = \sum_{i=1}^{l} p_i\alpha_i$$

for some positive integers p_i and denote

$$q^{2\rho} = \prod_{i=1}^{l} K_i^{p_i} \in U_q\mathfrak{g}. \tag{6.56}$$

Its commutation with the generators $X_i^{\pm}$ of $U_q\mathfrak{g}$ is given by

$$q^{2\rho}X_i^{\pm}q^{-2\rho} = q^{\pm(2\rho,\alpha_i)}X_i^{\pm}, \quad \forall i = 1, \dots, l.$$

As it is well known the square of the antipode in $U_q\mathfrak{g}$ is given by the following lemma.

Lemma 6.7. *For all* $x \in U_q\mathfrak{g}$

$$S^2(x) = q^{2\rho}xq^{-2\rho}.$$

For an arbitrary element $\nu = \sum_i m_i\alpha_i^\vee$ of the coroot lattice $Q^\vee$ of $\mathfrak{g}$ we set

$$q^{\nu} := (q^{m_1}, \dots, q^{m_l}) \in (\mathbb{C}^\times)^l \tag{6.57}$$

and consider the multiplicative character χ_{q^ν} of $\mathbb{C}_q[G]$. It is explicitly given by

$$\chi_{q^\nu}(c^\Lambda_{l,v}) = q^{(\nu,\mu)}\langle l, v\rangle, \quad l \in L(\Lambda)^*_{-\mu},$$

recall (3.17).

From Lemma 6.7 we deduce the following properties of S^2 in $\mathbb{C}_q[G]$.

Lemma 6.8. *(i) If* $v \in L(\Lambda)_\lambda$ *and* $l \in L(\Lambda)^*_{-\mu}$ *then*

$$S^2(c^\Lambda_{l,v}) = q^{2(\rho,\lambda-\mu)}c^\Lambda_{l,v}. \tag{6.58}$$

(ii) For all elements $w \in W$

$$ca_{\rho,w}a^*_{\rho,w} - \sum \chi_{q^{2(w\rho-\rho)}}(c_{(1)})a_{\rho,w}a^*_{\rho,w}S^2(c_{(2)}) \in I_w, \quad \forall c \in \mathbb{C}_q[G],$$

recall (3.20).

Proof. (i) By a straightforward computation, for all $x \in U_q\mathfrak{g}$

$$\langle S^2(c^\Lambda_{l,v}), x\rangle = \langle c^\Lambda_{l,v}, S^2(x)\rangle = \langle c^\Lambda_{l,v}, q^{2\rho}xq^{-2\rho}\rangle = q^{2(\rho,\lambda-\mu)}\langle c^\Lambda_{l,v}, x\rangle.$$

(ii) Combining part (i) with the identities (4.24) and (4.25) gives

$$c^\Lambda_{l,v}a_{\rho,w}a^*_{\rho,w} - q^{2(w\rho-\rho,\mu)}a_{\rho,w}a^*_{\rho,w}S^2(c^\Lambda_{l,v}) \in I_w, \quad \forall l \in L(\Lambda)^*_{-\mu}, v \in L(\Lambda).$$

which implies (6.58) in view of (3.11). □

Let us fix an element $w \in W$, a reduced decomposition $w = s_{i_1}\dots s_{i_n}$ of it, and an element $t \in (S^1)^l$. Consider the $(\mathbb{C}_q[G], *)$-module $V_{w,t}$. We will make use of the notation (4.29)

$$w_j = s_{i_{j+1}}\dots s_{i_n},\ j = 0, \dots, n-1, \quad w_n = 1$$

and of the basis $e_{k_1,\dots,k_n}$, $k_j \in \mathbb{N}$ of $V_{w,t}$ from (3.19).

Formula (4.31) implies that the space $V_{w,t}$ decomposes as a sum of weight subspaces with respect to the action of the commutative subalgebra of $\mathbb{C}_q[G]$ spanned by $a_{\Lambda,w}$, $\Lambda \in P_+$ (recall part (ii) of Proposition 4.1) as

$$V_{w,t} = \bigoplus_{\mu \in Q_+} \mathrm{span}\{e_{k_1,\dots,k_n} \mid \sum_{j=1}^{n} (k_j+1) w_j^{-1} \alpha_{i_j} = \mu\}. \tag{6.59}$$

All weight subspaces of $V_{w,t}$ are finite dimensional and we can identify the corresponding double restricted dual $V_{w,t}^{**}$ with $V_{w,t}$ as a vector space.

Part (ii) of Lemma 6.8 and the fact that the ideal I_w contains the annihilation ideal of $V_{w,t}$, see (3.21), imply that $\pi_{w,t}(a_{\rho,w} a_{\rho,w}^*)^{-1} \colon V_{w,t} \to V_{w,t}$ induces an isomorphism of the $\mathbb{C}_q[G]$-modules $V_{w,t}$ and $\chi \otimes V_{w,t}^{**}$. In view of Remark 6.4, $b_{w,t} = \pi_{w,t}(a_{\rho,w} a_{\rho,w}^*)^{-1}$ defines a quasi-balancing map for the $\mathbb{C}_q[G]$-module $\overline{V}_{w,t}$. Explicitly in the basis (3.19) of $V_{w,t}$, $\pi_{w,t}(a_{\rho,w} a_{\rho,w}^*)^{-1}$ acts diagonally by

$$\pi_{w,t}(a_{\rho,w} a_{\rho,w}^*)^{-1} . e_{k_1,\dots,k_n} = \prod_{j=1}^{n} q^{2(k_j+1)(w_j\rho, \alpha_{i_j})} e_{k_1,\dots,k_n}, \tag{6.60}$$

recall (4.31).

Define the set of quantum trace class operators in the $\mathbb{C}_q[G]$-module $\overline{V}_{w,t}$ by

$$\mathcal{B}_1^q(\overline{V}_{w,t}) = \mathcal{B}_1(\overline{V}_{w,t}) \pi_{w,t}(a_{\rho,w} a_{\rho,w}^*). \tag{6.61}$$

It is clear from (6.60) that $\pi_{w,t}(a_{\rho,w} a_{\rho,w}^*)$ is a compact operator and thus

$$\mathcal{B}_1^q(\overline{V}_{w,t}) \subset \mathcal{B}_1(\overline{V}_{w,t}).$$

Using Proposition 4.1, observe that

$$\pi_{w,t}(a_{2\rho,w} a_{2\rho,w}^*) = \pi_{w,t}(a_{\rho,w} a_{\rho,w}^*)^2. \tag{6.62}$$

Finally define the quantum quasi-trace functional $\mathrm{qtr}_{\overline{V}_{w,t}} \colon \mathcal{B}_1^q(\overline{V}_{w,t}) \to \mathbb{C}$ by

$$\mathrm{qtr}_{\overline{V}_{w,t}}(L) = \mathrm{const}_w \, \mathrm{tr}_{\overline{V}_{w,t}}(L \pi_{w,t}(a_{\rho,w} a_{\rho,w}^*)^{-1}) \tag{6.63}$$

where

$$\mathrm{const}_w = \prod_{\beta \in \Delta_+ \cap w^{-1}\Delta_-} (q^{(2\rho,\beta)} - 1). \tag{6.64}$$

Proposition 6.9. *The $\mathbb{C}_q[G]$-modules $\overline{V}_{w,t}$ are quasi-balanced with multiplicative characters $\chi_{2(w\rho-\rho)}$ and quasi-balancing morphisms $b_{w,t} = \pi_{w,t}(a_{\rho,w} a_{\rho,w}^*)^{-1}$. The space of quantum trace class operators in $\overline{V}_{w,t}$ and quantum quasi-trace morphisms*

$$\mathrm{qtr}_{\overline{V}_{w,t}} \colon \mathcal{B}_1^q(\overline{V}_{w,t}) \to \chi_{2(w\rho-\rho)}$$

are given by (6.61) *and* (6.63). *The morphisms $\mathrm{qtr}_{\overline{V}_{w,t}}$ are normalized by*

$$\mathrm{qtr}_{\overline{V}_{w,t}}(\pi_{w,t}(a_{2\rho,w} a_{2\rho,w}^*)) = 1. \tag{6.65}$$

To check (6.65) it is sufficient to check that

$$\mathrm{tr}_{\overline{V}_{w,t}}(\pi_{w,t}(a_{\rho,w} a_{\rho,w}^*)) = \prod_{\beta \in \Delta_+ \cap w^{-1}\Delta_-} (q^{(2\rho,\beta)} - 1)^{-1},$$

recall (6.62). This easily follows from (6.60) using the standard fact

$$\{w_j^{-1} \alpha_{i_j}\}_{j=1}^{n} = \Delta_+ \cap w^{-1} \Delta_- \tag{6.66}$$

in the notation of (4.29), see for instance [5].

Remark 6.10. Consider again the compact group K equipped with the standard Poisson structure, see the introduction and Section 5.6. Recall the notation $N_w = N \cap wN_- w^{-1}$ and $N_w^+ = N \cap wNw^{-1}$, $w \in W$, where N_- is the unipotent subgroup of G which is dual to N with respect to the fixed complex torus TA of G. The symplectic leaf $\mathcal{S}_w.t$ of K, considered as an AN homogeneous space under the dressing action, is isomorphic to AN/AN_w^+. We choose as a base point of $\mathcal{S}_w.t$ the point $\dot{w}.t$.

Denote by $\mu_{w,t}$ the Liouville volume form on the leaf $\mathcal{S}_w.t$. The diffeomorphisms

$$\mathcal{S}_w.t \cong AN/AN_w^+ \cong N_w \tag{6.67}$$

induce a measure dn_w on $\mathcal{S}_w.t$ from the Haar measure on N_w. The second one comes from the factorization $AN = N_w AN_w^+$. The measure dn_w will be normalized by

$$\int \left| a_{w,2\rho}|_{\mathcal{S}_w.t} \right|^2 dn_w = 1,$$

cf. Section 5.6.

The relation between the volume forms $\mu_{w,t}$ and dn_w on $\mathcal{S}_w.t$ was found by Lu [8]. It is given by

$$dn_w = \prod_{\beta \in \Delta_+ \cap w^{-1}\Delta_-} \left(\frac{(\rho, \beta)}{\pi} \right) \left| a_{w,\rho}|_{\mathcal{S}_w.t} \right|^{-2} \mu_{w,t}. \tag{6.68}$$

It is easy to compute that the measure dn_w on $\mathcal{S}_w.t$ transforms under the dressing action of $AN = K^*$ by

$$\delta_{an}\left(dn_w\right) = a^{2(\rho - w\rho)} dn_w.$$

The quantum quasi-trace morphisms

$$\mathrm{qtr}_{\overline{V}_{w,t}} \colon \mathcal{B}_1^q(\overline{V}_{w,t}) \to \chi_{2(w\rho - \rho)}$$

are quantum analogs of the measures dn_w on $\mathcal{S}_w.t$ and thus also of the Haar measures on the unipotent subgroups N_w of G. The traces in the modules $V_{w,t}$ can be considered as quantizations of the Liouville volume forms $\mu_{w,t}$ on the leaves $\mathcal{S}_w.t$. The relation (6.63) is a quantum version of Lu's relation (6.68).

6.4. **Tensor product properties of the quasi-balancing morphisms $b_{w,t}$.** When $w, w' \in W$ are such that $l(ww') = l(w) + l(w')$ the tensor product of $(\mathbb{C}_q[G], *)$-modules $V_{w,t} \otimes V_{w',t'}$ is again an irreducible $(\mathbb{C}_q[G], *)$-module, see Lemma 6.12 below. Here we discuss the relation between the corresponding quasi-balancing morphisms constructed in the previous subsection.

For an element $t = (t_1, \dots, t_l) \in (\mathbb{C}^\times)^l$ denote its j-th component by $(t)_j := t_j$. Define an action of the Weyl group W of $\mathfrak{g}$ on the torus $(\mathbb{C}^\times)^l$ by

$$(w(t))_i := \prod_j t_j^{m_{ij}} \quad \text{where} \quad w^{-1}\alpha_j^\vee = \sum_i m_{ij} \alpha_i^\vee.$$

It can be easily identified with the conjugation action of W on a complex torus of G. It is straightforward to check that

$$\chi_{w(t)}(c^\Lambda_{l,v}) = \prod_{i=1}^{l} t_i^{(\lambda, w^{-1}\alpha_i^\vee)} \langle l, v \rangle,$$

cf. (3.17).

Fix $w \in W$ and a reduced decomposition $w = s_{i_1} \ldots s_{i_n}$ of it. The representation space $V_{w,t}$, recall Theorem 3.3, is canonically identified with the vector space

$$V_w = V_{s_1} \otimes \ldots \otimes V_{s_n}$$

for all $t \in (\mathbb{C}^\times)^l$. (As earlier we will not show explicitly the dependence on the choice of a reduced decomposition of w.) Under this identification the basis (3.19) of $V_{w,t}$ corresponds to the basis

$$e_{k_1,\ldots,k_n} = e_{k_1} \otimes \ldots \otimes e_{k_n}, \quad n = l(w),\ k_1, \ldots, k_n \in \mathbb{N} \tag{6.69}$$

of V_w.

In the notation (4.29) define the linear operator $J_{w,t}$ in V_w acting diagonally in the above basis of V_w by

$$J_{w,t}.e_{k_1,\ldots,k_n} = \prod_{j=1}^{n} (w_{j-1}(t) w_j(t^{-1}))_{i_j}^{k_j+1} e_{k_1,\ldots,k_n}. \tag{6.70}$$

Lemma 6.11. *For all $w \in W$, and $t, t' \in (\mathbb{C}^\times)^l$ the operator $J_{w,t'}$ defines an isomorphism of the $\mathbb{C}_q[G]$-representations $\chi_{w(t')} \otimes \pi_{w,t}$ and $\pi_{w,t} \otimes \chi_{t'} \cong \pi_{w,tt'}$ in the natural identification of their representation spaces with V_w.*

Lemma 6.11 is checked directly in the case of $G = SL_2$ using the defining identities (3.15)–(3.16) for the $\mathbb{C}_q[SL_2]$-module π, see Section 3.3. This implies the lemma when w is a simple reflection and the general case is proved by induction on $l(w)$.

Lemma 6.12. *Let $w, w' \in W$ be such that $l(ww') = l(w) + l(w')$ and $t, t' \in (S^1)^l$. The linear operator $J_{w',(w')^{-1}(t)}$ induces the unitary equivalence of $(\mathbb{C}_q[G], *)$-modules*

$$\Pi_{w,t;w',t'} \colon \overline{V}_{w,t} \otimes \overline{V}_{w',t'} \to \overline{V}_{ww',(w')^{-1}(t)t'} \tag{6.71}$$

by identifying the spaces $V_{w,t} \otimes V_{w',t'} \cong V_w \otimes V_{w'} \cong V_{ww'} \cong V_{ww',(w')^{-1}(t)t'}$. (The product of two reduced decompositions of w and w' is used as a reduced decomposition of ww'.)

In the setting of Lemma 6.12 the $\mathbb{C}_q[G]$-module $\overline{V}_{ww',(w')^{-1}(t)t'}$ admits a quasi-balancing morphism constructed from the quasi-balancing morphisms $b_{w,t}$ and $b_{w',t'}$ for the modules $\overline{V}_{w,t}$ and $\overline{V}_{w',t'}$. It is given by the composition

$$\begin{aligned} (6.72)\quad & V_{ww',(w')^{-1}(t)t'} \xrightarrow{\Pi^{-1}_{w,t;w',t'}} V_{w,t} \otimes V_{w',t'} \xrightarrow{\mathrm{id} \otimes b_{w',t'}} \\ & V_{w,t} \otimes \chi_{q^{2(w'\rho-\rho)}} \otimes V^{**}_{w',t'} \xrightarrow{J^{-1}_{w,q^{2(w'\rho-\rho)}} \otimes \mathrm{id}} \chi_{q^{2w(w'\rho-\rho)}} \otimes V_{w,t} \otimes V^{**}_{w',t'} \xrightarrow{b_{w,t} \otimes \mathrm{id}} \\ & \chi_{q^{2w(w'\rho-\rho)}} \otimes \chi_{q^{2(w\rho-\rho)}} \otimes V^{**}_{w,t} \otimes V^{**}_{w',t'} \xrightarrow{\Pi^{**}_{w,t;w',t'}} \chi_{q^{2(ww'\rho-\rho)}} \otimes V^{**}_{ww',(w')^{-1}(t)t'}. \end{aligned}$$

The restricted duals of the modules $V_{w,t}$ and $V_{w',t'}$ are taken with respect to the weight space decomposition (6.59) for the commutative subalgebras of $\mathbb{C}_q[G]$ spanned by $a_{\Lambda,w}$ and $a_{\Lambda,w'}$, $\Lambda \in P_+$, respectively. Recall also the notation (6.57).

Proposition 6.13. *If $w, w' \in W$ are such that $l(ww') = l(w) + l(w')$ and $t, t' \in (S^1)^l$ then the quasi-balancing map for the $\mathbb{C}_q[G]$-module $\overline{V}_{ww',(w')^{-1}(t)t'}$ given by the composition (6.72) coincides with the quasi-balancing map $b_{ww',(w')^{-1}(t)t'}$.*

To prove Proposition 6.13 observe that in the natural identification of the representation spaces in (6.72) with $V_w \otimes V_{w'}$ the composition is simply

$$b_{w,t} J^{-1}_{w,q^{2(w'\rho-\rho)}} \otimes b_{w',t'}.$$

(We use again the product of two reduced decompositions of w and w' as a reduced decomposition of ww'.) Now the proposition easily follow from (6.60) and the following formula for the action of $J^{-1}_{w,q^{2(w'\rho-\rho)}}$ in the basis (6.69) of $V_{w,t}$ which is a direct consequence from (6.70)

$$J^{-1}_{w,q^{2(w'\rho-\rho)}}.e_{k_1,\dots,k_n} = \prod_{j=1}^{l(w)} q^{2(k_j+1)(w_j(w'\rho-\rho),\alpha_{i_j})} e_{k_1,\dots,k_n}.$$

This computation implies also the following connection between the spaces $\mathcal{B}_1^q(\overline{V}_{w,t})$, $\mathcal{B}_1^q(\overline{V}_{w',t'})$ and $\mathcal{B}_1^q(\overline{V}_{ww',(w')^{-1}(t)t'})$ when $l(ww') = l(w) + l(w')$.

Corollary 6.14. *If $L \in \mathcal{B}_1^q(\overline{V}_{w,t})$ and $L' \in \mathcal{B}_1^q(\overline{V}_{w',t'})$ in the setting of Proposition 6.13, then*

$$LJ_{w,q^{2(w'\rho-\rho)}} \otimes L' \in \mathcal{B}_1^q(\overline{V}_{ww',(w')^{-1}(t)t'})$$

and

$$\mathrm{qtr}_{\overline{V}_{ww',(w')^{-1}(t)t'}}(LJ_{w,q^{2(w'\rho-\rho)}} \otimes L') = \frac{\mathrm{const}_{ww'}}{\mathrm{const}_w \mathrm{const}_{w'}} \mathrm{qtr}_{\overline{V}_{w,t}}(L)\, \mathrm{qtr}_{\overline{V}_{w',t'}}(L')$$

where const_w *is given by* (6.64).

7. An application: quantum Harish-Chandra c-functions

Denote by 1 the identity element $(1,\dots,1)$ of the real torus $(S^1)^l$. According to (4.31) the linear operators $\pi_{w,1}(a_{\omega_i,w})$ in $\overline{V}_{w,1}$ are compact, selfadjoint with spectrum contained in $[0,\infty)$. For different values of i they mutually commute.

Hence for each $\lambda \in \mathfrak{h}$ we can define the linear operator in $\overline{V}_{w,1}$

$$d_{\lambda,w} = \prod_{i=1}^{l} \pi_{w,1}(a_{\omega_i,w})^{\lambda_i}$$

where $\lambda_i = (\lambda, \alpha_i^\vee)$, i.e. $\lambda = \sum \lambda_i \omega_i$. It is obvious that

$$d_{\lambda,w} = \pi_{w,1}(a_{\lambda,w}) \quad \text{when} \quad \lambda \in P_+ \subset \mathfrak{h} \tag{7.73}$$

and

$$d_{\lambda_1,w} d_{\lambda_2,w} = d_{\lambda_1+\lambda_2,w}, \quad \forall \lambda_1, \lambda_2 \in \mathfrak{h}. \tag{7.74}$$

Lemma 7.1. *The linear operator $d_{i\lambda+2\rho,w}$ in $\overline{V}_{w,1}$ is quantum trace class (belongs to $\mathcal{B}_1^q(\overline{V}_{w,1})$) if and only if*

$$\mathrm{Im}(\lambda,\beta) < 0, \quad \forall \beta \in \Delta_+ \cap w^{-1}\Delta_-.$$

Proof. The operator $d_{i\lambda+2\rho,w}$ in $\overline{V}_{w,1}$ belongs to $\mathcal{B}_1^q(\overline{V}_{w,1})$ if and only if

$$d_{i\lambda,w} \in \mathcal{B}_1(\overline{V}_{w,1})$$

because of (7.73), (7.74), and the the selfadjointness of $\pi_{w,1}(a_{\rho,w})$. The operator $d_{i\lambda,w}$ is diagonal in the orthonormal basis (3.19) of $\overline{V}_{w,1}$ and according to (4.31) acts by

$$d_{i\lambda,w}.e_{k_1,\dots,k_n} = \prod_{j=1}^{n} q^{-i(k_j+1)(w_j\lambda,\alpha_{i_j})} e_{k_1,\dots,k_n}, \tag{7.75}$$

recall the notation (4.29). It is clear that the linear operator $d_{i\lambda,w}$ in $\overline{V}_{w,1}$ is trace class if and only if $\mathrm{Re}(i\lambda, w_j^{-1}\alpha_{i_j}) > 0$ for $i = 1, \dots, n = l(w)$ which implies the statement because of (6.66). $\square$

Definition 7.2. The function

$$c^q_{w^{-1}}(\lambda) = \mathrm{qtr}_{\overline{V}_{w,1}}(d_{i\lambda+2\rho,w}) = \mathrm{tr}_{\overline{V}_{w,1}}(d_{i\lambda,w}) \tag{7.76}$$

in the domain $\{\lambda \in \mathfrak{h} \mid \mathrm{Im}(\lambda,\beta) < 0, \forall \beta \in \Delta_+ \cap w^{-1}\Delta_-\}$ will be called quantum Harish-Chandra c-function associated to the element w^{-1} of the Weyl group W of $\mathfrak{g}$.

Proposition 7.3. *For all $w \in W$ the quantum Harish-Chandra c-function $c^q_w(\lambda)$ is given by*

$$c^q_w(\lambda) = \prod_{\beta \in \Delta_+ \cap w\Delta_-} \frac{q^{(2\rho,\beta)} - 1}{q^{(i\lambda,\beta)} - 1}.$$

This proposition follows from (7.75) and (6.66) similarly to the proof of the normalization (6.65).

Remark 7.4. Proposition 7.3 is a quantum analog of the Harish-Chandra formula for the c-function in the case of complex simple Lie groups, generalized later by Gindikin and Karpelevich to arbitrary real reductive groups.

Recall the setting of Section 5.6 and Remark 6.10. Let dn_w denote the Haar measure on the unipotent subgroup N_w of G. The classical Harish-Chandra c-function associated to the element $w^{-1} \in W$ is given by the integral formula

$$c_{w^{-1}}(\lambda) = \int_{N_w} a_w(n)^{-(i\lambda+2\rho)} dn_w, \quad \lambda \in \mathfrak{h},\ \mathrm{Im}(\lambda,\beta) < 0,\ \forall \beta \in \Delta_+ \cap w^{-1}\Delta_-,$$

recall the definition (5.39) of the map $a_w \colon N \to A$. We refer to [3] for a detailed treatment of spherical functions and to [8] for an interpretation of the c-function in terms of the Poisson geometry of K, see in particular Example 2.8 in [8].

The linear operators $d_{i\lambda+2\rho}$ in the modules $\overline{V}_{w,1}$ can be thought of as quantizations of the pushforwards of the functions $a_w(n)^{-(i\lambda+2\rho)}$ on N_w to the symplectic leaves $\mathcal{S}_w$ by the dressing action, using the base points $\dot{w} \in \mathcal{S}_w$ (i.e. using the diffeomorphisms (6.67)). As was explained in Remark 6.10 the quantum quasi-traces $\mathrm{qtr}_{\overline{V}_{w,1}}$ in the $\mathbb{C}_q[G]$-modules $\overline{V}_{w,1}$ are quantizations of the pushforwards of the Haar measures on N_w to the symplectic leaves $\mathcal{S}_w$.

The classical Harish-Chandra formula

$$c_w(\lambda) = \prod_{\beta \in \Delta_+ \cap w\Delta_-} \frac{(2\rho,\beta)}{(i\lambda,\beta)}, \quad \lambda \in \mathfrak{h},\ \mathrm{Im}(\lambda,\beta) < 0,\ \forall \beta \in \Delta_+ \cap w^{-1}\Delta_-$$

is proved by induction on the length of w, see [3, Chapter IV, §6]. Lu [8] found that this argument is essentially based on the product formula (5.41) for the leaves $\mathcal{S}_w$.

Our computation relies on its quantum counterpart – the tensor product formula (3.18) for the representations $\pi_{w,t}$, cf. also Section 6.4.

References

[1] Chari, V. and Pressley, A.: *A guide to quantum groups.* Cambridge Univ. Press, 1994.

[2] Drinfeld, V. G.: *Quantum groups.* Proc. ICM, Berkeley, 1986, pp. 798–820. AMS, 1987.

[3] Helgason, S.: *Groups and geometric analysis.* Pure Appl. Math. 113, Acad. Press, 1984.

[4] Hodges, T. J. and Levasseur, T.: *Primitive ideals of* $\mathbb{C}_q[G]$. preprint 1992.

[5] Humphreys, J. E.: *Reflection groups and Coxeter groups.* Cambridge Stud. Adv. Math. 29, Cambridge Univ. Press, 1990.

[6] Joseph, A.: *Quantum groups and their primitive ideals.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer–Verlag, 1995.

[7] Korogodski, L. I. and Soibelman, Ya. S.: *Algebras of functions on quantum groups: Part I.* AMS Math. Surveys and Monographs 56, 1998.

[8] Lu, J.-H.: *Coordinates on Schubert cells, Kostant's harmonic forms, and the Bruhat Poisson structure on* G/B. Transform. Groups **4** (1999), no. 4, 355–374.

[9] Lu, J.-H. and Weinstein, A.: *Poisson Lie groups, dressing transformations, and Bruhat decompositions.* J. Diff. Geom. **31** (1990), no. 2, 501–526.

[10] Lusztig, G.: *Introduction to quantum groups.* Progr. Math. 110, Birkhuser, 1993.

[11] Reshetikhin, N. Yu. and Turaev, V. G.: *Ribbon graphs and their invariants derived from quantum groups.* Comm. Math. Phys. **127** (1990), no. 1, 1–26.

[12] Reshetikhin, N. Yu. and Turaev, V. G.: *Invariants of* 3*-manifolds via link polynomials and quantum groups.* Invent. Math. **103** (1991), no. 3, 547–597.

[13] Semenov-Tian-Shansky, M. A.: *Dressing transformations and Poisson group actions.* Publ. Res. Inst. Math. Sci. **21** (1985), no. 6, 1237–1260.

[14] Soibelman, Ya. S.: *The algebra of functions on a compact quantum group and its representations.* St. Petersburg Math. J. **2** (1991), 193–225.

[15] Soibelman, Ya. S.: Selected topics in quantum groups In: *Infinite analysis, Part B* (Kyoto, 1991), 859–887, Adv. Ser. Math. Phys., 16, World Sci. Publ., 1992.

[16] Soibelman, Ya. S. and Vaksman, L. L.: *An algebra of functions on the quantum group* $SU(2)$. Funct. Anal. Appl. **22** (1988), no. 3, 170–181.

[17] Soibelman, Ya. S. and Vaksman, L. L.: On some problems in the theory of quantum groups. In: *Representation theory and dynamical systems,* 3–55, Adv. Soviet Math., 9, AMS 1992.

[18] Woronowicz, S. L.: *Compact matrix pseudogroups.* Comm. Math. Phys. **111** (1987), no.4, 613–665.

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720, U.S.A.

E-mail address: reshetik@math.berkeley.edu

Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720, U.S.A.

E-mail address: yakimov@math.berkeley.edu